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THE SPREAD OF REGULAR SPACES

Judy ROITMAN
Department of Mathematics, Wellesley College, Wellesley, Massachusetts 02181, U.S.A.

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Under either CH or not-SH, there exists a 0-dimensional Hausdorff space of countable spread which is not the union of a hereditarily separable and a hereditarily Lindelöf space. Under not-SH$+2^\omega>\aleph_{\omega_1}$, there exists a 0-dimensional Hausdorff space of spread $\aleph_{\omega_1}$ which has no discrete subspace of cardinality $\aleph_{\omega_1}$.

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spread Lindelöf
separable Luzin

0. Introduction

In [10] and [9], Hausdorff spaces were found which, respectively

(a) had a spread of ω but was not the union of a hereditarily separable and a hereditarily Lindelöf space

(b) had a spread of $\aleph_{\omega_1}$ but no discrete subspace of cardinality $\aleph_{\omega_1}$ (assuming $2^\omega\geqslant\aleph_{\omega_1}$).

Unfortunately, these examples had the property that any subspace of the same cardinality as the space was not regular. So the question remaining was whether regular spaces with these properties could be found.

The answer is: in certain models of set theory, yes.

Specifically, let A be the statement: $\exists$ a 0-dimensional Hausdorff space satisfying (a); and B the statement: $\exists$ a 0-dimensional Hausdorff space satisfying (b). Then, where CH is the continuum hypothesis and SH the statement that there exist no Suslin lines:

(1) CH$\to$A

(2) not-SH$\to$A

(3) not-SH$+2^\omega\geqslant\aleph_{\omega_1}\to$B.

Can absolute examples be found? Juhász and Hajnal show in [5], e.g., that GCH$\to$not-B, so no absolute example of a space satisfying B exists. It is also well-known that a regular space of countable spread which is not hereditarily separable contains an L-space; and a regular space of countable spread which is not hereditarily Lindelöf contains an S-space. Thus an absolute example of a space satisfying A would contain a proof of the existence of S and L space — a consummation which some may devoutly wish, but which this paper does not attempt.

1. Preliminaries

a. The *spread* of a space is the supremum of the cardinalities of its discrete subspaces. The *weight* of a space is the infimum of the cardinalities of its bases. We note that a first-countable space of cardinality κ has weight $\leq \kappa$.

b. A space is *hereditarily separable* (abbreviated hs) iff every subspace contains a countable dense subset. It is *hereditarily Lindelöf* (abbreviated hL) iff every open cover of a subspace contains a countable subcover of that subspace. We note that if a space is either hs or hL, its spread is countable.

c. An *S-space* is a regular Hausdorff hs non-Lindelöf space. An *L-space* is a regular Hausdorff hL non-separable space. Any 0-dimensional space which is S or L contains a subset which, under a weaker topology, is still a 0-dimensional S or L space and has weight ω_1.

d. A space has the *countable chain condition* (abbreviated ccc) iff every family of disjoint open sets is countable. We note that a space of countable spread has the ccc.

e. The following characterization will be used implicitly everywhere. A space is *right-separated* (respectively *left-separated*) iff it can be well-ordered so that every initial segment is open (closed). A space is not hL (not hs) iff it has an uncountable right-separated (left-separated) subspace [7]. A right-separated space has no uncountable Lindelöf subspaces and is hs iff it has no uncountable discrete subspaces; similarly, a left-separated space has no uncountable separable subspaces and is hL iff it has no uncountable discrete subspaces. For convenience, and contrary to common practice of everyone including the author, during the rest of this paper we assume that S-spaces are right-separated and L-spaces are left-separated; further, given an S or L space on an ordinal, the order of the separation agrees with the usual ordinal order.

f. Let E be any set, I an index set, and $\mathcal{A} = \{A_i : i \in I\}$ be a family of subsets of E. Then $\mathcal{A}$ is an *independent family on* $\mathscr{E}$ iff for any finite collection $i_0, \ldots, i_k, i_{k+1}, \ldots, i_n$ of distinct elements of I, the intersection

$$\bigcap_{j \leq k} A_{i_j} \cap \bigcap_{k < j \leq n} (E - A_{i_j})$$

has the same cardinality as E. A theorem of Hausdorff says that if E is infinite, there is an independent family on E of cardinality $2^{\operatorname{card}(E)}$.

g. Finally, let $X = \bigcup_{\alpha < \kappa} X_\alpha$, where the X_α's are mutually disjoint. Then Y is a *partial transversal* of X iff every $\operatorname{card}(Y \cap X_\alpha) \leq 1$.

2. The basic construction

To avoid notational complexities, an ordinal will be considered as the set of its predecessors and can have a topology on it which has absolutely nothing to do with the usual ordinal topology. For the rest of this section $\mathcal{T}$ is fixed to be any 0-dimensional first-countable Hausdorff topology on ω_1. To every point α in ω_1 is thus associated a strictly descending ω-sequence $\{u'_{i,\alpha} : i < \omega\}$ of clopen sets in $\mathcal{T}$ which form a neighborhood basis for α. Let $u_{i,\alpha} = u'_{i,\alpha} - u'_{i+1,\alpha}$. Then $u'_{i,\alpha} = \{\alpha\} \cup \bigcup_{j \geq i} u_{j,\alpha}$ is the description of the neighborhood basis which will be used, and we emphasize that, for each α, the $u_{i,\alpha}$'s are clopen, mutually disjoint, non-empty, and none of them contain the point α.

For $\alpha < \omega_1$, let X_α be a 0-dimensional Hausdorff space of weight $\leq 2^\omega$ under the topology $\mathcal{T}_\alpha$, and let $\mathcal{V}_\alpha$ be a clopen basis for X_α of cardinality $\leq 2^\omega$. We let $\mathcal{A}_\alpha$ be an independent family on ω whose elements are indexed by the elements of $\mathcal{V}_\alpha$; thus $\mathcal{A}_\alpha = \{A_v : v \in \mathcal{V}_\alpha\}$.

Letting X be the disjoint union of the X_α's, we proceed to put a topology $\mathcal{T}^*$ on X which is Hausdorff and 0-dimensional, and such that the relative topology $\langle X_\alpha, \mathcal{T}^*\rangle = \langle X_\alpha, \mathcal{T}_\alpha\rangle$, for every α.

A sub-basis is defined and then explained. Thus let $v \in \mathcal{V}_\alpha$, $i \in \omega$. Then $v \cap X_\beta = \emptyset$ if $\beta \neq \alpha$, and $A_v \in \mathcal{A}_\alpha$. The following two sets are open:

(i) $v \cup \bigcup \{X_\beta : \beta \in u_{j,\alpha}, j \in A_v$ and $j > i\}$. Call this set $w_{\alpha,v,i}$.

(ii) $(X_\alpha - v) \cup \bigcup \{X_\beta : \beta \in u_{j,\alpha}$ for some $j \in (\omega - A_v)$ and $j \geq i\}$. Call this set $w_{\alpha, X_\alpha - v, i}$.

What we do, then, is look at a basic clopen neighborhood v of X_α and some $u'_{i,\alpha} = \{\alpha\} \cup \bigcup_{j \geq i} u_{j,\alpha}$. We use A_v to pick out infinitely many of the $u_{j,\alpha}$'s for $j \geq i$. Then we add to v entire levels X_β whose indices fall in the $u_{j,\alpha}$'s which we've picked. And we add to $X_\alpha - v$ entire levels X_β whose indices fall in the $u_{j,\alpha}$'s which we've not picked.

Let $\mathcal{T}^*$ be the topology generated by this sub-basis. Clearly each $\langle X_\alpha, \mathcal{T}^*\rangle = \langle X_\alpha, \mathcal{T}_\alpha\rangle$. We also note that if $u \in \mathcal{T}$, then $\bigcup_{\alpha \in u} X_\alpha$ is open. Using this we show that $\langle X, \mathcal{T}^*\rangle$, which from now on we simply call X, is both Hausdorff and 0-dimensional.

Claim 1. *X is Hausdorff.*

Proof. Suppose $x \in X_\alpha$, $y \in X_\beta$, $\alpha \neq \beta$. Then $\exists$ clopen $u \in \mathcal{T}$, $\alpha \in u$, $\beta \notin u$. Hence $\bigcup_{\gamma \in u} X_\gamma$ and $\bigcup_{\gamma \notin u} X_\gamma$ are open and separate x and y. If, on the other hand, $x \neq y$ and both $x, y \in X_\alpha$, let v be clopen, $v \in \mathcal{T}_\alpha$, $x \in v$, $y \notin v$. Then $w_{\alpha,v,i}$ and $w_{\alpha,(X_\alpha - v),i}$ are both open and separate x and y.

Claim 2. *X is 0-dimensional.*

Proof. We need only show that our sub-basis is clopen. Without loss of generality we consider only $w_{\alpha,v,i}$. So suppose $x \notin w_{\alpha,v,i}$. Then either:

(i) $x \in X_\alpha$, $x \notin v$, and so $w_{\alpha,(X_\alpha - v),i}$ is a neighborhood containing x and disjoint from $w_{\alpha,v,i}$; or

(ii) $x \in X_\gamma$, $\gamma \notin u'_{i,\alpha}$, in which case $\bigcup_{\beta \notin u'_{i,\alpha}} X_\beta$ is a neighborhood of x disjoint from $w_{\alpha,v,i}$; or

(iii) $x \in X_\gamma$, $\gamma \in u_{j,\alpha}$ for $j \geq i$, in which case $\bigcup_{\beta \in u_{j,\alpha}} X_\beta$ is a neighborhood of x disjoint from $w_{\alpha,v,i}$.

3. The basic lemmas

The question now is when the construction of Section 2 has the properties needed for A or B. This is answered in a series of lemmas. Lemmas 1 and 2, while substantially stated and proved in [9] and [10], will for convenience be stated and sketched here. The reader will note that the hypothesis of Lemma 4 specifically requires the construction of Section 2.

Lemma 1. *Suppose $X = \bigcup_{\alpha<\omega_1} X_\alpha$ is a regular Hausdorff space where the X_α's are mutually disjoint, and suppose either*

(i) *every uncountable partial transversal of X is an S-space and every X_α is an L-space; or*

(ii) *every uncountable partial transversal of X is an L-space and every X_α is an S-space.*

Then X satisfies A.

Proof. Since every uncountable subset of X either has uncountable intersection with some X_α or contains an uncountable partial transversal, the spread must be countable. And if we divide X into two pieces, then one piece both contains an uncountable partial transversal and an uncountable subset of some X_α. So this piece can be neither hs or hL.

Lemma 2. *Suppose $X = \bigcup_{\alpha<\omega_1} X$ is a regular Hausdorff space, the X_α's are mutually disjoint, each X_α is a discrete subspace of cardinality $\aleph_\alpha$ and each partial transversal of X has spread ω. Then X satisfies B.*

Proof. Any subspace Y of X of cardinality $\aleph_{\omega_1}$ must contain an uncountable partial transversal which has spread ω and thus is not discrete. So Y is not discrete.

The next step is to find candidates for a topology $\mathcal{T}$ on ω_1 so that Lemmas 1 and 2 have a possibility of being satisfied by the construction of Section 2.

Definition 3. A *Luzin space* is a Hausdorff space with no isolated points and no uncountable closed nowhere dense sets (i.e. every uncountable closed set contains an open set).

The important facts about Luzin spaces for our purposes are: every Luzin space is hereditarily Lindelöf (Tall [13]); a regular Luzin space is 0-dimensional (an unpublished observation of Kunen); hence a regular non-separable Luzin space has a subspace which is a Luzin L-space.

Lemma 4. *Let $\mathcal{T}$ be a regular first-countable Luzin topology on ω_1. Then if $X = \bigcup_{\alpha<\omega} X_\alpha$, $\mathcal{T}$, $\mathcal{T}^*$, are as in Section 2, no uncountable partial transversal Y of X is discrete. Furthermore, if $\mathcal{T}$ is also an L-space topology and Y is an uncountable partial transversal of X, then Y is an L-space.*

Proof. An uncountable partial transversal Y of X will be Hausdorff, 0-dimensional and not separable by the results of Section 2. We show that Y is not discrete. (By our convention that L-spaces are assumed left-separated, this will prove both parts of the lemma.)

Suppose Y is discrete. Let $Q_Y = \{\gamma : Y \cap X_\gamma \neq \emptyset\}$. For $\gamma \in Q_Y$, let w^γ be an open set in $\mathcal{T}^*$ such that $w^\gamma \cap Y = X_\gamma \cap Y$; i.e. the w^γ's witness the discreteness of Y. Hence for infinitely many $k \in \omega$, $\bigcup \{X_\alpha : \alpha \in u_{k,\gamma}\} \subset w^\gamma$. Let B_γ be the set of such k.

We now show that if u is open non-empty in $\mathcal{T}$, for some open $v \subset u$, $v \cap Q_Y = \emptyset$. Thus the closure of Q_Y is nowhere dense in $\mathcal{T}$, but since Q_Y is uncountable, this contradicts $\mathcal{T}$ being Luzin.

So let u be open non-empty, $u \in \mathcal{T}$. If $u \cap Q_Y = \emptyset$, we have nothing to prove. Otherwise, let $\gamma \in u \cap Q_Y$. Then for some n, $u'_{n,\gamma} \subset u$. Let $k \in B_\gamma$, $k > n$. Then $u_{k,\gamma} \subset u$ and $u_{k,\gamma} \subset Q_Y = \emptyset$. Lemma 4 is thus proved.

For completeness we state the following; while for readability only the barest hint of proof is given.

Lemma 5. *Let $\mathcal{T}$ be the first-countable S-space on ω_1 constructed by Juhász, Kunen and Rudin* [8]. *Then with a bit of extra care in the choice of the $\mathcal{A}_\alpha$'s, if X, X_α, $\mathcal{T}_\alpha$, $\mathcal{T}^*$ are as in Section 2, any uncountable partial transversal of X is an S-space.*

Sketch of proof. Identify the reals as a set with ω_1. The key to the construction in [8] was that the topology of the reals was refined so that: (i) the closure of a countable set in the new topology differed by at most a countable set from its closure in the old topology; and (ii) in constructing the neighborhoods of a point α you were concerned with at most countably many countable sets $B_{n,\alpha}$ which had the point α in their respective closures in the old topology, making sure that a new neighborhood of α hit each $B_{n,\alpha}$ in an infinite set; and (iii) each countable infinite $B \subset \omega_1$ had an associated γ_B so that if $\alpha > \gamma_B$ then B would be $B_{n,\alpha}$ for some n.

To prove Lemma 5, the family $\mathcal{A}_\alpha$ is constructed so that the new neighborhoods of $\mathcal{T}^*$ will respect (ii); that is, if w is a neighborhood in $\mathcal{T}^*$ of a point in X_α, then for each n, $\{\gamma \in B_{n,\alpha} : X_\gamma \subset w\}$ is infinite. This gives the pleasant result that if Y is an uncountable transversal in X, Q_Y is as in Lemma 4, and $B \subset Q_Y$ is dense in Q_Y

under $\mathcal{T}$, then $\{Y \cap X_\gamma : \gamma \in B\}$ is actually dense in all but a countable subset of Y, thus proving Y is separable which was the only interesting thing to prove.

Finally, we put all these lemmas together in

Corollary 6. (i) *Let $\langle X, \mathcal{T}^* \rangle$ be as in Lemma 4, where each X_α is a 0-dimensional S-space of weight ω_1. Then X satisfies A.*

(ii) *Let $\langle X, \mathcal{T}^* \rangle$ be as in Lemma 5, where each X_α is a 0-dimensional L-space of weight ω_1. Then X satisfies A.*

(iii) *Let $\langle X, \mathcal{T}^* \rangle$ be as in Lemma 4, where $2^\kappa \geq \aleph_{\omega_1}$ and each X_α is a discrete space of cardinality $\aleph_\alpha$. Then X satisfies B.*

4. Consistency and generalizations

The final question, then, is what set-theoretic hypotheses give us the necessary spaces to satisfy the lemmas? A brief answer would be: lots of them.

For Lemma 4 a first-countable Luzin L-space is needed; for Lemma 5 a specific S-space that we know exists under CH; for Corollary 6(i) any S-space (since every S-space has a 0-dimensional uncountable subspace); and for 6(ii) any 0-dimensional L-space. Where can we find these spaces?

Under CH, many 0-dimensional S and L spaces exist (see, for a partial survey, [11]). Hence, by Corollary 6(ii), CH$\rightarrow$A. In [1], van Douwen, Tall, and Weiss construct a first-countable Luzin L-space under CH, and so CH$\rightarrow$A also follows from 6(i).

SH is the statement that no Suslin lines exist, where a Suslin line is a non-separable ccc linearly ordered space. Every Suslin line is first countable and has a Luzin subspace. Hence not-SH gives us what we need for Lemma 4. It is well-known that not-SH is consistent with "the continuum as large as you wish". In particular, not-SH $+ 2^\omega \geq \aleph_{\omega_1}$ is consistent; by 6(iii), not-SH $+ 2^\omega \geq \aleph_{\omega_1} \rightarrow$ B. Furthermore, Rudin [12] used not-SH to get an S-space; so by 6(i), not-SH$\rightarrow$A.

A particular handy model of set theory also gives us several ways of establishing both A and B, namely a Cohen-generic extension adding κ Cohen reals, where $\mathrm{cf}(\kappa) > \omega$. By work of Solovay and Tennenbaum, not-SH holds in such a model, so the preceding paragraph applies; furthermore, the set of Cohen reals is Luzin in the topology of the reals, so if $\kappa \geq \aleph_{\omega_1}$, B also follows from 6(iii).

Finally, we indicate briefly how to generalize to other cardinals. Let $B(\lambda, \gamma)$ be the statement: B with λ substituted for $\aleph_{\omega_1}$, where λ has cofinality γ; let $A(\kappa)$ be the statement A with κ, κ-separable, and κ-Lindelöf substituted where appropriate. Assume κ is regular, and note that a κ^+-Suslin line has a κ-Luzin subspace which can be used in the obvious analogue of Lemma 4. Thus we have

$$\text{not-}\kappa^+\text{-SH} + 2^\omega \geq \lambda \rightarrow B(\lambda, \kappa^+);$$

while by noting further that under GCH there exists a 0-dimensional κ-S-space for every κ (Juhász, Hajnal), we conclude

$$\mathrm{GCH} + \text{not-}\kappa^{+}\text{-SH} \rightarrow A(\kappa).$$

The hypothesis of the first statement is well-known to be consistent; that the second statement holds in L is a classic result of Jensen.

References

[1] E. van Douwen, F. Tall and W. Weiss, CH entails the existence of non-metrizable hereditarily Lindelöf spaces with point-countable bases, preprint.

[2] A. Hajnal and I. Juhász, A consistency result concerning hereditarily α-separable spaces, Indag. Math. 35 (1973).

[3] A. Hajnal and I. Juhász, On first-countable non-Lindelöf hereditarily separable spaces, Math. Inst. Hung. Acad. 711.

[4] A. Hajnal and I. Juhász, Discrete subspaces of topological spaces, Indag. Math. 29 (1967).

[5] A. Hajnal and I. Juhász, Discrete subspaces of topological spaces II, Indag, Math. 31 (1969).

[6] R. Jensen, The fine structure of the constructible hierarchy, Annals of Math. Logic 4 (1972).

[7] I. Juhász, Cardinal functions in topology (Math. Centrum, Amsterdam, 1971).

[8] I. Juhász, K. Kunen and M.E. Rudin, Another hereditarily separable non-Lindelöf space, preprint.

[9] J. Roitman, Attaining the spread at cardinals which are not strong limits, Pac. Journ. Math. 57 (1975).

[10] J. Roitman, A space of small spread without the usual properties, Proc. AMS 43 (1974).

[11] M.E. Rudin, Lectures on set-theoretic topology, AMS, 1975.

[12] M.E. Rudin, A normal hereditarily separable non-Lindelöf space, Ill. J. Math. 16 (1972).

[13] F. Tall, The density topology, preprint.